Thomas rotation (or Thomas precession)

an elegant, exact 3-dimensional derivation
with simple explanations of all preliminaries

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ABSTRACT. The Thomas rotation arising from two successive Lorentz transformations with velocities in different directions is derived using an elegant notation. The general form of a Lorentz boost and the relativistic combination of two velocities in different directions are explained first .

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1. *Introduction*

In special relativity, a Lorentz transformation without spatial rotation or inversion of coordinate axes, between inertial frames in constant relative motion, is known as a *Lorentz boost*. The Thomas rotation is a spatial rotation which appears when two Lorentz boosts in different directions are combined. It was originally calculated in the infinitesimal case [1]. For completeness, the necessary preliminary results are derived in Part 1. The Thomas rotation is dealt with in Part 2.

Elegance and simplicity are increased by the use of bold type for vectors and tensors. One of the pioneers of this was Milne [2]. Kennedy [3] used Lorentz matrices in a similar way to the present paper, and partially exploited the advantages of the vector-tensor notation. He did, however, obtain an explicit formula for the rotation, as did Ungar [4]. This paper contents itself with a precise geometrical description of the rotation.

The bold type notation and the results in Sections 3, 4 appeared in an earlier paper [5], but the derivation of the Thomas rotation was not given there

Part 1

2. *The vector-tensor notation*

4×4 matrices may be written very neatly using vectors or vector-like triples and 3×3 tensor or tensor-like arrays to specify submatrices. Such vector- and tensor-like arrays are here used formally, without reference to any transformation properties in a 3-dimensional space. Hence the terminology in this paper will be simplified to 'vector' and 'tensor'. However, within a given inertial frame, the arrays are indeed Cartesian vectors and tensors.

If the vector **a** has components (a_1, a_2, a_3), then the matrix $[\mathbf{a}] = [a_1, a_2, a_3]$. In particular, $\mathbf{0} = (0, 0, 0)$ gives $[\mathbf{0}] = [0\ 0\ 0]$. If **A** is a second-rank tensor with components A_{jk}, then $[\mathbf{A}]$ is the matrix with components $[\mathbf{A}]_{jk} = A_{jk}$.

Then the 4×4 matrix below, with components a_{jk}, is conveniently expressed in terms of submatrices. $[\mathbf{B}]$ is the appropriate 3×3 matrix, $[\mathbf{d}] = [a_{14}\ a_{24}\ a_{34}]$ where a prime denotes the transpose of a matrix, and $[\mathbf{e}] = [a_{41}\ a_{42}\ a_{43}]$. (*Note: matrix braces are displayed defectively.)*

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} [\mathbf{B}] & [\mathbf{d}]' \\ [\mathbf{e}] & a_{44} \end{bmatrix}$$

Matrix mutliplication and Cartesian vector/tensor inner/outer products follow the same rules, so

$$[\mathbf{a}][\mathbf{b}]' = [\mathbf{a}\cdot\mathbf{b}], \quad [\mathbf{a}][\mathbf{A}] = [\mathbf{a}\cdot\mathbf{A}], \quad [\mathbf{A}][\mathbf{B}] = [\mathbf{A}\cdot\mathbf{B}], \quad [\mathbf{a}]'[\mathbf{b}] = [\mathbf{ab}], \quad [\mathbf{A}][\mathbf{a}]' = [\mathbf{A}\cdot\mathbf{a}]'$$

(*Note: in some expressions below, square brackets are algebraic bracketing.*)

3. *Lorentz boost with velocity **u***

In this Section we derive the vectorial equations for a Lorentz boost. Let a Lorentz boost with velocity $(u, 0, 0)$ transform Cartesian coordinates and time (x, y, z, t) into (x', y', z', t'). This means that the frame with primed coordinates is moving with velocity u along the x-axis relative to the frame with unprimed coordinates. Hence if

$$\boxed{\gamma(u) \equiv (1 - u^2/c^2)^{-1/2}}$$

we have

$$x' = \gamma(u)(x - ut) \tag{1}$$

$$y' = y, \quad z' = z \tag{2}$$

$$t' = \gamma(u)(t - xu/c^2) \tag{3}$$

To find the vectorial form of these equations, let the relative velocity be $\mathbf{u}$ in a general direction and let $\mathbf{n}$ be the unit vector in the direction of $\mathbf{u}$. Then $|\mathbf{u}| = u$, $\mathbf{u} = u\mathbf{n}$. Let $\mathbf{U}$ be the unit tensor in 3 dimensions, $\mathbf{U} = \mathrm{diag}(1, 1, 1)$, and write

$$\mathbf{W} = \mathbf{U} - \mathbf{nn}$$

Note that $\mathbf{n}\cdot\mathbf{W} = \mathbf{W}\cdot\mathbf{n} = \mathbf{0}$.

Choose the positive x-axis in the direction $\mathbf{n}$. With $\mathbf{r} = (x, y, z)$, $\mathbf{r}' = (x', y', z')$, we have $x = \mathbf{r}\cdot\mathbf{n}$, $(0, y, z) = \mathbf{r}\cdot\mathbf{W}$, and similarly for primed coordinates. Then (1)–(3) give

$$\begin{aligned} \mathbf{r}'\cdot\mathbf{n} &= \gamma(u)(\mathbf{r}\cdot\mathbf{n} - ut) \\ \mathbf{r}'\cdot\mathbf{W} &= \mathbf{r}\cdot\mathbf{W} \\ t' &= \gamma(u)(t - \mathbf{r}\cdot\mathbf{n}u/c^2) \\ &= \gamma(u)(t - \mathbf{r}\cdot\mathbf{u}/c^2) \end{aligned}$$

Then

$$\begin{aligned} \mathbf{r}' &= \mathbf{r}'\cdot\mathbf{nn} + \mathbf{r}'\cdot\mathbf{W} \\ &= \gamma(u)(\mathbf{r}\cdot\mathbf{nn} - u\mathbf{n}t) + \mathbf{r}\cdot\mathbf{W} \end{aligned}$$

giving finally

$$\boxed{\begin{aligned} \mathbf{r}' &= \boldsymbol{\Lambda}(\mathbf{u})\cdot\mathbf{r} - \gamma(u)\mathbf{u}t & (4) \\ t' &= \gamma(u)(-\mathbf{u}\cdot\mathbf{r}/c^2 + t) & (5) \end{aligned}}$$

where

$$\boldsymbol{\Lambda}(\mathbf{u}) \equiv (\gamma(u) - 1)\mathbf{nn} + \mathbf{U}$$

These equations are independent of the choice of coordinates.

The 4×4 matrix $L(\mathbf{u})$ of this transformation, such that $[[\mathbf{r}'], t']' = L(\mathbf{u})[[\mathbf{r}], t]'$ (note the two different meanings of a prime here!), is

$$L(\mathbf{u}) = \begin{bmatrix} [\boldsymbol{\Lambda}(\mathbf{u})] & -\gamma(u)[\mathbf{u}]' \\ -\gamma(u)[\mathbf{u}]/c^2 & \gamma(u) \end{bmatrix}$$

It is easy to verify that

$$\boldsymbol{\Lambda}(-\mathbf{u}) = \boldsymbol{\Lambda}(\mathbf{u}), \qquad \boldsymbol{\Lambda}^{-1}(\mathbf{u}) = \gamma^{-1}(u)\mathbf{nn} + \mathbf{W}$$

$$\mathbf{u}\cdot \boldsymbol{\Lambda}(\mathbf{u}) = \boldsymbol{\Lambda}(\mathbf{u})\cdot\mathbf{u} = \gamma(u)\mathbf{u}, \qquad \mathbf{u}\cdot\boldsymbol{\Lambda}^{-1}(\mathbf{u}) = \boldsymbol{\Lambda}^{-1}(\mathbf{u})\cdot\mathbf{u} = \gamma^{-1}(u)\mathbf{u} \tag{6}$$

Since L(–**u**) is the inverse transformation, we have

$$\boxed{\begin{aligned} \mathbf{r} &= \boldsymbol{\Lambda}(\mathbf{u})\cdot\mathbf{r}' + \gamma(u)\mathbf{u}t' \quad (7) \\ t &= \gamma(u)(\mathbf{u}\cdot\mathbf{r}'/c^2 + t') \quad (8) \end{aligned}}$$

4. *General composition of velocities*

The composition of two velocities in different directions gives the longest calculations required to derive the Thomas rotation. Let a frame F′ have velocity **u** relative to a frame F, and let the velocity of some point P measured in F′ be **v.** Then the velocity of P measured in F will be denoted by $\mathbf{u} \oplus \mathbf{v}$. For simplicity, put

$$\mathbf{u}_1 = \mathbf{u} \oplus \mathbf{v}, \quad \mathbf{u}_2 = \mathbf{v} \oplus \mathbf{u}$$
$$\gamma_u = \gamma(u), \quad \gamma_v = \gamma(v), \quad \gamma_1 = \gamma(u_1)$$

Starting at (**0**, 0) in F, F′, let P move to ($\mathbf{u}_1$T, T) in F and (**v**T′, T′) in F′. Substituting these coordinates in (4), (8) gives by (6)

$$\mathbf{v}T' = (\boldsymbol{\Lambda}(\mathbf{u})\cdot\mathbf{u}_1 - \gamma_u\mathbf{u})T = \boldsymbol{\Lambda}(\mathbf{u})\cdot(\mathbf{u}_1 - \mathbf{u})T$$
$$T = \gamma_u(\mathbf{u}\cdot\mathbf{v}/c^2 + 1)T'$$

Hence

$$\boxed{\mathbf{u}_1 = \mathbf{u} + \gamma_u^{-1}\boldsymbol{\Lambda}^{-1}(\mathbf{u})\cdot\mathbf{v}/(1 + \mathbf{u}\cdot\mathbf{v}/c^2)} \tag{9}$$

We now need to find a formula for γ_1. For convenience, put $q = \mathbf{u}\cdot\mathbf{v}/c^2$.
Since **n**=**u**/u we have, by definition

$$\boldsymbol{\Lambda}^{-1}(\mathbf{u})\cdot\mathbf{v} = (\gamma_u^{-1} - 1)q\mathbf{u}/(u^2/c^2) + \mathbf{v}$$

Express u^2/c^2 in terms of γ_u and calculate $(1 + q)^2(1 - u_1^2/c^2)$ using (6). The terms in q, q^2 disappear and we obtain

$$\boxed{\gamma_1 = \gamma_u\gamma_v(1 + \mathbf{u}\cdot\mathbf{v}/c^2)} \tag{10}$$

Because $\mathbf{u}_2$ is obtained from $\mathbf{u}_1$ by interchanging **u**, **v**, it is clear from the symmetry of (10) that $\gamma_2 = \gamma_1$. Hence

$$\boxed{u_1 = u_2}$$

Substituting the coordinates of P in (4) instead of (1) gives immediately

$$\boxed{\mathbf{u}_1 = (\gamma_v/\gamma_1)[\boldsymbol{\Lambda}(\mathbf{u})\cdot\mathbf{v} + \gamma_u\mathbf{u}]} \tag{11}$$

Part 2

5. *General Thomas rotation*

Theorem 1.

$$\mathrm{L}(\mathbf{u})\mathrm{L}(\mathbf{v}) = \mathrm{L}(\mathbf{u} \oplus \mathbf{v})\mathrm{R}(\mathbf{u}, \mathbf{v}) \tag{12}$$

$$= \mathrm{R}(\mathbf{u}, \mathbf{v})\mathrm{L}(\mathbf{v} \oplus \mathbf{u}) \tag{13}$$

where R(**u**, **v**) *is the matrix carrying* $\mathbf{v} \oplus \mathbf{u}$ *into* $\mathbf{u} \oplus \mathbf{v}$ *and leaving time coordinates unchanged*

Proof: By direct calculation using (11) for $\mathbf{u}_1$, $\mathbf{u}_2$

$$\mathrm{L}(\mathbf{u})\mathrm{L}(\mathbf{v}) = \begin{pmatrix} [\mathbf{M}] & -\gamma_1[\mathbf{u}_1]' \\ -\gamma_1[\mathbf{u}_2]/c^2 & \gamma_1 \end{pmatrix} \tag{14}$$

where

$$\mathbf{M} = \mathbf{\Lambda}(\mathbf{u})\cdot\mathbf{\Lambda}(\mathbf{v}) + \gamma_u\gamma_v\,\mathbf{uv}/c^2$$

By direct calculation again

$$\mathrm{L}(-\mathbf{u}_1)\mathrm{L}(\mathbf{u})\mathrm{L}(\mathbf{v}) = \begin{pmatrix} [\mathbf{\Lambda}(-\mathbf{u}_1)] & \gamma_1[\mathbf{u}_1]' \\ \gamma_1[\mathbf{u}_1]/c^2 & \gamma_1 \end{pmatrix} \begin{pmatrix} [\mathbf{M}] & -\gamma_1[\mathbf{u}_1]' \\ -\gamma_1[\mathbf{u}_2]/c^2 & \gamma_1 \end{pmatrix} \tag{15}$$

$$= \begin{pmatrix} [\mathbf{R}] & [\mathbf{0}]' \\ [\mathbf{0}] & 1 \end{pmatrix}$$

where **R** is some tensor whose meaning will be found indirectly.

> Note that the entry [**0**] arises from using (9) for $\mathbf{u}_1$ (noticing that $\mathbf{\Lambda}$ and its inverse are symmetic), and (11) for $\mathbf{u}_2$, with (6), in
>
> $\mathbf{u}_1\cdot\mathbf{M} - \gamma_1\mathbf{u}_2$
>
> $= [\mathbf{u} + \gamma_u^{-1}\mathbf{v}\cdot\mathbf{\Lambda}^{-1}(\mathbf{u})/(1 + \mathbf{u}\cdot\mathbf{v}/c^2)]\cdot[\mathbf{\Lambda}(\mathbf{u})\cdot\mathbf{\Lambda}(\mathbf{v}) + \gamma_u\gamma_v\,\mathbf{uv}/c^2] - \gamma_u\,[\mathbf{\Lambda}(\mathbf{v})\cdot\mathbf{u} + \gamma_v\mathbf{v}]$
>
> $= \mathbf{0}$.

6. *Meaning of **R***

Since the left hand side of (15) leaves invariant $\mathbf{r}^2 - c^2t^2$, [**R**] must be an orthogonal matrix. This matrix is also proper, because when **u**, **v** are parallel, only the simple form of the Lorentz transformations occur, for which there is no rotation and [**R**] = I in this case.

Hence (15) establishes (12) apart from the meaning of the rotation R(**u**, **v**).

To identify [**R**], rewrite (12) as

$$\mathrm{L}(\mathbf{u}_1) = \mathrm{L}(\mathbf{u})\mathrm{L}(\mathbf{v})\mathrm{R}^{-1}(\mathbf{u}, \mathbf{v}) \tag{16}$$

Since [**R**] is orthogonal, $[\mathbf{R}]^{-1} = [\mathbf{R}]'$ and the first three columns of (16) are

(17)
$$\begin{bmatrix} [\mathbf{\Lambda}(\mathbf{u}_1)] & \\ -\gamma_1[\mathbf{u}_1]/c^2 & \end{bmatrix} = \begin{bmatrix} [\mathbf{M}] & \text{²} \\ -\gamma_1[\mathbf{u}_2]/c^2 & \text{²} \end{bmatrix} \begin{bmatrix} [\mathbf{R}]' \\ [\mathbf{0}] \end{bmatrix}$$

where ² denotes elements not contributing to this product. The fourth row of (17) gives $[\mathbf{u}_1] = [\mathbf{u}_2]\,[\mathbf{R}]'$, and the transpose of this is

(18)
$$\boxed{[\mathbf{u}_1]' = [\mathbf{R}]\,[\mathbf{u}_2]'}$$

Hence the rotation R(**u**, **v**) is the Thomas rotation as defined in Theorem 1.

This proves (12). Note also that the first row of (18) is, since the tensor **Λ** is symmetric,

(19)
$$[\mathbf{\Lambda}(\mathbf{u}_1)] = [\mathbf{M}][\mathbf{R}]' = [\mathbf{R}][\mathbf{M}]'$$

To prove (13), interchange **u**, **v** in (12) so that

$$\mathrm{L}(\mathbf{u}_2)\mathrm{R}(\mathbf{v}, \mathbf{u}) = \mathrm{L}(\mathbf{v})\mathrm{L}(\mathbf{u})$$

By definition, $\mathrm{R}(\mathbf{v}, \mathbf{u}) = \mathrm{R}^{-1}(\mathbf{u}, \mathbf{v})$, and applying R(**u**, **v**) on the left gives, by (14)

$$\begin{aligned} \mathrm{R}(\mathbf{u}, \mathbf{v})\mathrm{L}(\mathbf{u}_2)\mathrm{R}^{-1}(\mathbf{u}, \mathbf{v}) &= \mathrm{R}(\mathbf{u}, \mathbf{v})\mathrm{L}(\mathbf{v})\mathrm{L}(\mathbf{u}) \\ &= \begin{bmatrix} [\mathbf{R}] & [\mathbf{0}]' \\ [\mathbf{0}] & 1 \end{bmatrix} \begin{bmatrix} [\mathbf{M}]' & -\gamma_1[\mathbf{u}_2]' \\ -\gamma_1[\mathbf{u}_1]/c^2 & \gamma_1 \end{bmatrix} \\ &= \begin{bmatrix} [\mathbf{R}][\mathbf{M}]' & -\gamma_1[\mathbf{R}][\mathbf{u}_2]' \\ -\gamma_1[\mathbf{u}_1]/c^2 & \gamma_1 \end{bmatrix} \\ &= \mathrm{L}(\mathbf{u}_1) \end{aligned}$$

by (19), (18). Then

$$\mathrm{R}(\mathbf{u}, \mathbf{v})\mathrm{L}(\mathbf{u}_2) = \mathrm{L}(\mathbf{u}_1)\mathrm{R}(\mathbf{u}, \mathbf{v})$$

This is equivalent to (13), and Theorem 1 is proved.

References

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