ASYMPTOTIC METHODS

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Abstract. Here we look to provide an introduction to the theory of asymptotic analysis, starting from the basic definitions, and ultimately focusing on perturbation theory and the methods of approximating differential equations. Two of the most common methods in singular perturbation theory shall be presented matched asymptotic expansions and multiple scales with fully worked examples to demonstrate the theory.

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1. Introduction

Many of the functions that arise from everyday problems cannot easily be evaluated exactly, particularly those defined in terms of integrals or differential equations. In these situations we usually have two options. We can use computers to seek complicated numerical solutions or we can look to construct an analytical approximation to the solution using asymptotic expansions. Asymptotic methods have particular importance in many areas of applied mathematics, with the physical problems studied in fluid dynamics providing the main motivation for much of the important development in the subject’s history.

One of the oldest and most famous asymptotic results is Stirling’s formula, used to approximate \( n! \) for large values of \( n \),

\[
  n! \sim \left( \frac{n}{e} \right)^n \sqrt{2\pi n}
\]

where \( \sim \) is used to denote that two functions are asymptotically, or approximately, equivalent. It was Henri Poincaré who introduced the term asymptotic expansion during an 1886 paper published in Acta Mathematica[14], studying irregular integrals of linear equations. He began that paper by analysing another of Stirling’s series: his series for the logarithm of the Euler Gamma Function.

The subject of approximating is best studied on a case-by-case basis as there are a very wide range of methods necessary for different situations, with little possibility of presenting a general rule. This project will focus on the methods applicable to problems presented as differential equations, particularly the areas of regular and singular perturbation theory. Following the lead of most publications the theory will be introduced amid examples where the methods of matched asymptotic expansions and multiple scales will both be considered in detail.

The overall aim of this project is to introduce the reader to some of the basic concepts of asymptotic analysis and to lead up to presenting two of the main methods of perturbation theory in detail. The intended readership are honours level students with some knowledge of ordinary and partial differential equations but who, like myself before beginning my research, may have very little idea what an asymptotic expansion is and who may not have met the order notation before.

1.1. Notation. To study this subject we must first introduce the Order Notation: A function \( f(x) \) is said to be "big-O" \( g(x) \), written \( f(x) = O(g(x)) \), as \( z \to z_0 \) if

\[
  \exists K, \delta > 0 \text{ such that } |f| \leq K|g| \text{ when } z_0 < z < \delta
\]

for some constants \( K \) and \( \delta \).

The use of the equality symbol is unfortunate and misleading as it is correctly read "\( f(x) \text{ is big-O } g(x) \)", not "\( f(x) \text{ equals big-O } g(x) \)".

A useful alternative form for this is \( f(x) = O(g(x)) \) if

\[
  \lim_{z \to z_0} \frac{f(z)}{g(z)} = C
\]

for C a finite non-zero constant. Essentially, if a non-zero finite limit exists.

Similarly we have little-o: \( f(x) = o(g(x)) \), as \( z \to z_0 \) if

\[
  \forall K, \exists \delta > 0 \text{ such that } |f| \leq K|g| \text{ when } z_0 < z < \delta
\]

or alternatively if

\[
  \lim_{z \to z_0} \frac{f(z)}{g(z)} = 0
\]
In summary $O$ states that a function is of smaller or equal order to another, whilst $o$ states that the order is strictly less. Thus $f(x) = o(g(x)) \implies f(x) = O(g(x))$.

1.2. Definitions.

**Asymptotic Equivalence:** The aim of asymptotic approximation is to find a function that is asymptotically equivalent to the solution of the given problem. Two functions $f$ and $g$ are asymptotically equivalent, written $f(x) \sim g(x)$ as $x \to x_0$, if

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = 1$$

**Asymptotic Sequence:** A sequence $\{\phi_n(z)\}$ for $n=1,2,...$ is called an asymptotic sequence if $\forall n$,

$$\phi_{n+1}(z) = o(\phi_n(z)) \text{ as } z \to z_0.$$

The order of every term in an asymptotic sequence is less than the order of the previous one.

**Asymptotic Expansion:** Poincaré’s original definition states that for $\phi_n(z)$ an asymptotic sequence as $z \to z_0$, $\sum_{n=1}^{N} a_n \phi_n(z)$ is an asymptotic expansion (or asymptotic series) of $f(z)$ if $\forall N$

$$f(z) = \sum_{n=1}^{N} a_n \phi_n(z) + o(\phi_N(z)) \text{ as } z \to z_0$$

For all values of $n$ the functions $\phi_n(z)$ are known as the gauge functions.

1.3. Introductory Examples. One of the most useful tools for obtaining asymptotic expansions is Taylor’s theorem which is often used within more complicated methods. As an example consider the exponential function $e^z$.

The Taylor series for $z \to 0$ is well known as

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + ... \text{ valid for all } z \in \mathbb{C}.$$

From the definition this is clearly a valid asymptotic series for $z \to 0$ as the order of the terms is gradually decreasing, hence for a 3 term asymptotic expansion for $e^z$ we can write

$$e^z \sim 1 + z + \frac{z^2}{2!}.$$

This is known as 3 term expansion or an expansion to $O(z^3)$ (to order $z^3$) taken from the highest order of the asymptotic variable the series is expanded to.

1.3.1. Example. Now to work through our first example demonstrating some asymptotic methods, consider the quadratic equation

$$x^2 + \epsilon x - 1 = 0. \quad (1.1)$$

The two solutions are dependent on parameter $\epsilon$ ($0 < \epsilon \ll 1$) so the asymptotic expansion is performed with respect to $\epsilon$. It is expected that the solutions will have the form

$$x \sim x_0 + \epsilon x_1 + \epsilon^2 x_2 + ... \quad (1.2)$$

Substituting (1.2) into (1.1) gives, after rearranging,

$$x_0^2 - 1 + \epsilon(2x_0x_1 + x_0) + \epsilon^2(x_1^2 + 2x_0x_2 + 2x_0x_1) + ... = 0. \quad (1.3)$$

Now all the parts of the equation must be balanced to give 0 so a comparison is made between parts of the same order with respect to $\epsilon$. 

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Firstly we compare the parts that do not involve $\epsilon$, of order 1 $O(1)$:

$$x_0^2 - 1 = 0$$

$\therefore$  $x_0 = \pm 1$.

And then, $O(\epsilon)$:

$$2x_0x_1 + x_0 = 0$$

$\therefore$  $x_1 = -\frac{1}{2}$.

This process can be continued to find as many terms of the expansion as needed. The two terms we have form an asymptotic expansion for problem (1.1) up to $O(\epsilon)$:

$$x \sim \pm 1 - \frac{1}{2} \epsilon.$$  (1.4)

Often this expansion would be written with $+O(\epsilon^2)$ at the end to indicate the order of the truncated terms.

Using the quadratic formula the exact solutions to the problem are found to be

$$x = \frac{1}{2}(\pm \sqrt{\epsilon^2 + 4} - \epsilon),$$

Expanding this to as a Taylor series for $\epsilon \to 0$ we get

$$x = \pm 1 - \frac{1}{2} \epsilon + O(\epsilon^2)$$

confirming that for the expansion found in (1.4) is a good approximation to problem (1.1).

It must be noted that the use of $O(1)$ and $O(\epsilon)$ here when equating the terms from equation (1.3) is somewhat ambiguous. Strictly speaking all the terms in the equation are $O(1)$ since $\epsilon \ll 1$, what we actually want to include are the terms which are $O(1)$ but not $o(1)$; hence the terms which are exactly of order 1 with respect to $\epsilon$. This apparent misuse is common notation however so we will continue to use $O$ in these situations to refer terms which are of the specified order.

2. Perturbation Theory

One of the main uses of asymptotic analysis is to provide approximations to differential equations that cannot easily be solved explicitly. For simplicity we shall consider only 2nd order ordinary differential equations but most of the techniques used are easily expanded to problems of higher order or to partial differential equations.

The following is a general 2nd order differential equation for $y(x, \lambda)$, a function of $x$ and $\lambda$

$$\frac{d^2y}{dx^2} + p(x, \lambda)\frac{dy}{dx} + q(x, \lambda)y = r(x, \lambda).$$

The independent variable here is $x$, with respect to which all differentiation and integration is applied. $\lambda$ and any other variables upon which the solution of $y$ could depend on are known as physical parameters and no differentiation or integration is carried out with respect to them.

The variable with respect to which we study the asymptotic behaviour is known as the asymptotic variable. In classical asymptotic analysis the asymptotic variable is taken as the independent variable of the differential equation. In perturbation theory the asymptotic behaviour is studied with respect to a small physical parameter, usually denoted by $\epsilon$. Modern asymptotic approximation theory can deal with cases where the
asymptotic behaviour is studied relative to both the independent variable and a physical parameter together, but this lies beyond the scope of this project.

The point in the domain around which the asymptotic behaviour is studied is known as the asymptotic accumulation point, which was denoted by $z_0$ in the definitions in section 1. The most common differential equation problems where we might look for an approximation are those of perturbation theory, where the accumulation point is $\epsilon = 0$. It is those that we focus on in this project.

Perturbation theory deals with problems that contain a small parameter conventionally denoted by $\epsilon$ and solutions are sought as $\epsilon$ approaches 0. The simple quadratic problem given by (1.1) containing $\epsilon$ as a coefficient of $x$ is an example of a perturbation problem. Perturbation theory can be split into regular and singular forms but the differences between the two will be dealt with when they arise in the coming examples.

2.1. Regular Perturbation. The general method with perturbation problems is to seek an expansion with respect to the asymptotic sequence $\{1, \epsilon, \epsilon^2, \ldots\}$ as $\epsilon \to 0$. The regular (or Poincaré) expansion is then

$$u(\epsilon, x) \sim U_0(x) + \epsilon U_1(x) + \epsilon^2 U_2(x) + \ldots \quad \text{as} \quad \epsilon \to 0$$

for gauge functions $U_0, U_1, \ldots$ which we will determine. An example of a quadratic regular perturbation problem was given in (1.1) earlier to introduce asymptotic expansions but now we shall consider a physical problem showing how perturbations can arise in differential equations.

2.1.1. Example. Consider the initial value problem

$$\frac{d^2 y}{d\tau^2} = -\epsilon \frac{dy}{d\tau} - 1 \quad y(0) = 0, \quad \frac{dy}{d\tau}(0) = 1.$$  \hspace{1cm} (2.1)

The equation here represents a projectile motion problem where air friction taken into account. $\epsilon = \frac{kv_0}{mg}$ where $k$ is the coefficient of air friction, $g$ is the gravitational acceleration, $m$ the object’s mass and $v_0$ the initial velocity [15].

The procedure is the same as was used for the polynomial equation in (1.1), namely assuming a solution expanded in terms of $\epsilon$ by Taylor:

$$y(\tau) = y_0(\tau) + \epsilon y_1(\tau) + \epsilon^2 y_2(\tau) + \ldots$$  \hspace{1cm} (2.2)

This is now substituted in the differential equation and initial conditions (2.1) to determine functions $y_0, y_1$ and $y_2$ and give a 3 term expansion. Of course this can be carried out to find as many terms of the expansion as necessary but in practical situations only a small number of terms are usually needed. Substituting gives, after rearranging:

$$\frac{d^2 y_0}{d\tau^2} + 1 + \epsilon \left( \frac{d^2 y_1}{d\tau^2} + \frac{dy_0}{d\tau} \right) + \epsilon^2 \left( \frac{d^2 y_2}{d\tau^2} + \frac{dy_1}{d\tau} \right) + O(\epsilon^3) = 0$$

$$y_0(0) + \epsilon y_1(0) + \epsilon^2 y_2(0) + O(\epsilon^3) = 0$$

$$\frac{dy_0}{d\tau}(0) - 1 + \epsilon \frac{dy_1}{d\tau}(0) + \epsilon^2 \frac{dy_2}{d\tau}(0) + O(\epsilon^3) = 0$$

The next step is then to equate to zero all the terms of each order of $\epsilon$: 
\( O(1) : \quad \frac{d^2 y_0}{d\tau^2} + 1 = 0 \quad \text{where} \quad y_0(0) = 0 \quad \frac{dy_0}{d\tau}(0) - 1 = 0, \)
\( O(\epsilon) : \quad \frac{d^2 y_1}{d\tau^2} + \frac{dy_0}{d\tau} = 0 \quad \text{where} \quad y_1(0) = 0 \quad \frac{dy_1}{d\tau}(0) = 0, \)
\( O(\epsilon^2) : \quad \frac{d^2 y_2}{d\tau^2} + \frac{dy_1}{d\tau} = 0 \quad \text{where} \quad y_2(0) = 0 \quad \frac{dy_2}{d\tau}(0) = 0. \)

Solving these equations gives:

\[
y_0(\tau) = \tau - \tau^2/2, \quad y_1(\tau) = -\tau^2/2 + \tau^3/6, \quad y_2(\tau) = \tau^3/6 - \tau^4/24.
\]

Now putting these into the expansion (2.2) gives an approximation to \( O(\epsilon^2) \):

\[
y(\tau) \sim (\tau - \frac{\tau^2}{2!}) + \epsilon(-\frac{\tau^2}{2!} + \frac{\tau^3}{3!}) + \epsilon^2(\frac{\tau^3}{3!} - \frac{\tau^4}{4!}). \quad (2.3)
\]

The differential equation in (2.1) can be solved to find the exact solution

\[
y(\tau) = \frac{(1 + \epsilon)}{\epsilon^2} (1 - e^{-\epsilon\tau}) - \frac{\tau}{\epsilon}.
\]

We can expand this as a Taylor series to provide an expansion comparable to our perturbation solution (2.3) to check its accuracy. Firstly recall the Taylor expansion for \( 1 - e^{-z} = z - \frac{z^2}{2} + \frac{z^3}{3!} - \ldots \) which is used in obtaining

\[
y(\tau) = (\tau - \frac{\tau^2}{2!}) + \epsilon(-\frac{\tau^2}{2!} + \frac{\tau^3}{3!}) + \epsilon^2(\frac{\tau^3}{3!} - \frac{\tau^4}{4!}) + O(\epsilon^3).
\]

Noticeably this is identical to the solution obtained in (2.3) using the perturbation methods above.

It appears then that finding an asymptotic expansion is simply a case of assuming a well-known form and substituting it into the problem equation to give an approximate solution. Of course this was an extremely basic example but the fundamental principle is the same wherever regular perturbation techniques can be used. In the next section on singular perturbation theory we will discuss areas where regular perturbation fails.

The example shows that regular perturbation problems can be found in Newtonian Mechanics problems considering projectile motion. Van Dyke[6] deals in great detail with the physical problems from fluid mechanics where perturbation techniques are useful, a couple of examples given are the Janzen-Rayleigh expansion at low Mach number and the inviscid flow of fluids about solid shapes such as a circle or cylinder. Another interesting example comes from Deeba and Xie’s[9] solution of the Van der Pol Oscillator, which crosses a wide scientific spectrum with it’s possible applications; arising in models of electrical circuit theory, neuroscience and geology for example.

2.2. Singular Perturbation. A perturbation problem is said to be singular when the regular methods produce an expansion that fails, at some point, to be valid over the complete domain. To introduce a singular perturbation type problem, again we look at a simple polynomial equation similar to the one used in the introductory section. Consider the problem

\[
\epsilon x^2 + 2x - 1 = 0
\]

and notice the similarities to (1.1) but note the important difference - here \( \epsilon \) is a coefficient of the leading order term \( x^2 \). Following the standard method by inserting the regular expansion \( x \sim x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots \)
and equating the coefficients gives as the solution:

\[ x \sim \frac{1}{2} - \epsilon \frac{1}{8} + \ldots \]

Clearly the regular method has failed - the problem is quadratic so we should have two solutions but only one has been produced. In many cases this situation is easy to spot by setting \( \epsilon = 0 \) to give the unperturbed equation. When \( \epsilon \) is the leading order term’s sole coefficient the order of the equation is reduced in the unperturbed equation. In this example to a linear equation with only one solution.

There are several types of singular perturbation problems that all require different methods to tackle them. We will consider two of the most common and widely applicable methods - Matched asymptotic expansions and the method of Multiple Scales.

When \( \epsilon \) is a multiplier of the highest derivative or leading term of a polynomial equation it is known as a boundary layer problem or occasionally a matching problem, the reasons for which will become clear in the next section.

## 3. Matched Asymptotic Expansions

The method of matched asymptotic expansions has its roots in Ludwig Prandtl’s boundary layer theory, developed in 1905 whilst studying the flow of a viscous incompressible fluid past a wall or body. In physical terms a boundary layer is the layer of fluid at the very edge of a flow, against the containing surface for example an aircraft wing or the banks of a river. The thin area here exhibits properties vastly different from the rest of the field.

### 3.1. Introductory Example

To introduce the method we will consider a simple differential equation problem with associated boundary conditions for \( 0 < x < 1 \) featured in Holmes [3]:

\[
\begin{align*}
\epsilon \frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} + 2y &= 0 \\
y(0) &= 0 \\
y(1) &= 1
\end{align*}
\]  \hspace{1cm} (3.1)

This is immediately recognisable as a boundary layer problem from the \( \epsilon \) coefficient multiplying the 2nd order derivative.

#### 3.1.1. Outer Solution

We will start by assuming that the solution has a regular Taylor expansion for \( \epsilon \), so consider the familiar equation

\[ y(x) \sim y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x) + \ldots \]  \hspace{1cm} (3.2)

Substituting this into 3.1 gives:

\[ \epsilon(\frac{d^2 y_0}{dx^2} + \epsilon \frac{d^2 y_1}{dx^2} + \ldots) + 2(\frac{dy_0}{dx} + \epsilon \frac{dy_1}{dx} + \epsilon^2 \frac{dy_2}{dx} + \ldots) + 2(y_0 + \epsilon y_1 + \epsilon^2 y_2) = 0. \]  \hspace{1cm} (3.3)

And so the term independent of \( \epsilon \), O(1) is:

\[ \frac{dy_0}{dx} + y_0 = 0 \]

with the general solution

\[ y_0(x) = Ae^{-x} \]  \hspace{1cm} (3.4)

for \( A \) an arbitrary constant. This raises the first difficulty: with only 1 arbitrary constant both boundary conditions cannot be satisfied. What this means is that (3.2) and (3.4) cannot provide a valid solution over the whole interval \( 0 \leq x \leq 1 \) and this is where Prandtl’s boundary theory comes into play. It is now assumed
that a boundary layer exists at either end of the domain \([0,1]\) within which the properties of the solutions are vastly different, preventing solution (3.4) being valid over the complete interval. For now we assume that this boundary layer occurs at \(x = 0\) and so we will have more than one solution. Inside the boundary layer about \(x = 0\) we will have an \textit{inner} or boundary layer \textit{solution} and over the rest of the domain we have the \textit{outer solution}. Hence (3.4) will be known as first term of the outer solution and it satisfies the second of the two boundary conditions \(y(1) = 1\).

Using this boundary condition (3.4) can be solved exactly as

\[ y_0(x) = e^{1-x}. \]  

This is our 1 term approximation to the solution outside the boundary layer.

3.1.2. \textit{Boundary Layer}. To deal with the boundary layer at \(x = 0\) introduce a \textit{boundary layer coordinate} to rescale the problem within the layer;

\[ X = \frac{x}{\delta(\epsilon)}, \]  

where in this example \(\delta(\epsilon)\) is expected to be some power of \(\epsilon\). X is also known as the stretching coordinate because under the transformation \(x \to X\) with \(X\) taken to be fixed, the region becomes much larger as \(\epsilon \to 0\). Let \(Y(X)\) denote the solutions to the problem when using the boundary layer coordinate and so from (3.6) and the chain rule we get

\[ \frac{d}{dx} = \frac{dX}{dx} \frac{d}{dX} = \frac{1}{\delta(\epsilon)} \frac{d}{dX} \]  

and

\[ \frac{d^2}{dx^2} = \frac{d}{dx} \left\{ \frac{1}{\delta(\epsilon)} \frac{d}{dX} \right\} = \frac{1}{\delta(\epsilon)^2} \frac{d^2}{dX^2} \]  

Using these new coordinates the initial problem (3.1) becomes

\[ \frac{\epsilon}{\delta(\epsilon)^2} \frac{d^2Y}{dX^2} + \frac{2}{\delta(\epsilon)} \frac{dY}{dX} + 2Y = 0 \]  

with \(Y(0) = 0\)

and the task is to determine the function \(\delta(\epsilon)\). The problem with the initial equation (3.1) was that the coefficient of the highest derivative in the ODE was small in comparison to the rest \((\epsilon < < 2)\). Now we must compare all the coefficients to choose \(\delta(\epsilon)\) in such a way that maintains the first coefficient as the largest.

There are three possibilities in equating to zero the three coefficients \(\frac{\epsilon}{\delta(\epsilon)^2}, \frac{2}{\delta(\epsilon)}, 2\):

**Case 1:** If we assume then that the first and second coefficients are of the same order,

\[ \frac{\epsilon}{\delta(\epsilon)^2} \sim \frac{2}{\delta(\epsilon)} \]

we get \(\delta(\epsilon) = \epsilon\) resulting in the equation becoming:

\[ \frac{1}{\epsilon} \frac{d^2Y}{dX^2} + \frac{2}{\epsilon} \frac{dY}{dX} + 2Y = 0. \]

So now the coefficient of the highest derivative is of the same order as the coefficient of the first derivative and is larger that that of the \(Y\) term, since \(\frac{1}{\epsilon} > 2\) as \(\epsilon \to 0\). This case therefore gives a possible \(\delta(\epsilon)\) but for completeness check the other two remaining cases.
Case 2:

\[
\frac{\epsilon}{\delta(\epsilon)^2} \sim 2
\]

\[
=> \delta(\epsilon) = \epsilon^2
\]

which would make the equation:

\[
\frac{d^2Y}{dX^2} + \frac{2}{\epsilon^2} \frac{dY}{dX} + 2Y = 0.
\]

But now the coefficient of the first derivative \(\frac{2}{\epsilon^2}\) is much larger than the other two, so this case is not possible.

Case 3:

\[
\frac{2}{\delta(\epsilon)} \sim 2
\]

\[
=> \delta(\epsilon) = 1
\]

and hence the problem would become

\[
\epsilon \frac{d^2Y}{dX^2} + 2 \frac{dY}{dX} + 2Y = 0
\]

which gives the highest derivative a coefficient of order \(\epsilon\), again much smaller than the coefficients of O(1) for the other terms. So from Case 1 the correct choice of boundary layer coordinate will be

\[
X = x/\epsilon,
\]

and the equation is now

\[
\frac{1}{\epsilon} \frac{d^2Y}{dX^2} + \frac{2}{\epsilon} \frac{dY}{dX} + 2Y = 0. \quad (3.10)
\]

The appropriate expansion within the boundary layer shall be

\[Y(X) \sim Y_0(X) + \epsilon Y_1(X) + \epsilon^2 Y_2(X) + ...\]

Substituting this into (3.10) gives

\[
O\left(\frac{1}{\epsilon}\right): \frac{d^2Y_0}{dX^2} + \frac{dY_0}{dX} + 2Y_0 = 0
\]

Now we solve for \(Y_0, Y_1,\ldots\) by balancing the terms in the equation by order of \(\epsilon\):

\[
O\left(\frac{1}{\epsilon}\right): \frac{d^2Y_0}{dX^2} + 2 \frac{dY_0}{dX} = 0
\]

The general solution of which is

\[Y_0(X) = A(1 - e^{-2X}) \quad (3.11)\]

for arbitrary constant \(A\).

This is the solution to the problem within the boundary region at \(x = 0\) and we notice that the constant \(A\) cannot be determined by either of the boundary conditions. \(A\) must be determined then by the matching process.
3.1.3. Matching. The important idea is to realize that both the inner and outer expansions are approximations to the same function. Hence where the two meet both expansions should provide a valid and equal result. At this stage we already know the outer expansion explicitly but thus far our inner expansion depends on an unknown constant. The matching process is used to evaluate the constants in the boundary layer expansion and it also plays an important part in forming the composite expansion.

Essentially, as $Y_0$ leaves the boundary layer ($X \rightarrow \infty$) it must be equal to $y_0$ as it comes in to the boundary layer, when ($x \rightarrow 0$). From this we obtain that

$$\lim_{X \rightarrow \infty} Y_0 = \lim_{x \rightarrow 0} y_0$$

and hence

$$\lim_{X \rightarrow \infty} A(1 - e^{-2X}) = e^{1-0}$$

which gives $A = e^1$. And so we finally have the first term of our inner expansion

$$Y_0(X) = e^1 - e^{1-2X}.$$  \hspace{1cm} (3.12)

3.1.4. Composite Expansion. Having now obtained expressions for the first terms of both the inner and the outer expansions they must now be matched together to obtain one composite expansion that approximates the solution over the whole domain $[0, 1]$. To get the composite expansion the inner and outer expansions are simply added together and the common limit found in (3.12) is subtracted, otherwise it would be included twice in the overlapping region. So our solution to $O(1)$ is

$$y_{comp} \sim y_0(x) + Y_0\left(\frac{x}{e}\right) - e^1$$

$$\sim e^{1-x} - e^{1-\frac{2x}{e}}.$$  \hspace{1cm} (3.14)

![Figure 1. Exact solution vs Approximation](image)
For comparison figure 1 shows the exact solution plotted against our composite expansion (3.14) with \( \epsilon = 0.01 \), using MAPLE. The exact solution is plotted using a solid line and our approximation is the dashed line, but even with only a one-term expansion the two are almost indistinguishable.

3.1.5. Further Terms. With the bulk of the theory now out the way we can return to the differential equations obtained using the outer and inner expansions and repeat the same balancing and matching processes to calculate a second term for each expansion. From here a second term can be added into the composite expansion to give a more accurate approximation.

The expansions matched this time will be slightly more complicated than in the first-order case so we need to revise our matching method. Previously we asserted that the limits \( \lim_{X \to \infty} Y(X) \) and \( \lim_{x \to 0} y(x) \) were equal. Essentially this is a simplification of the intermediate matching principle formulated by Saul Kaplun, which introduces the concept of another region in the problem, essentially an overlap region where the inner and outer expansions meet and where both will be valid.

We define a new intermediate variable, valid over an overlap domain, to bridge between the inner and outer regions already defined. In conventional notation this interval is written \([\eta_0(\epsilon), \eta_1(\epsilon)]\) and in elementary terms it is the region where both the inner and outer approximations are valid.

Figure 2. Overlap region

Figure 2 shows a comprehensible illustration (similar to Holmes’ diagram [3]) of how the boundary layer comes into play. The inner approximation is valid in the region around \( x = 0 \), valid at \( \epsilon \) which is close to 0 and valid across the newly defined intermediate domain \([\eta_0, \eta_1]\). By definition the outer approximation is also valid across this interval and over the remaining section of the problem domain, up to \( x = 1 \).

Figure 2 also gives a good image of how process for obtaining a composite expansion works. The inner and outer expansions are added together and the solution over the intermediate domain is subtracted. Quite simply if we did not subtract the intermediate solution it would be counted twice as this region is already approximated by both the inner and the outer expansions.

Proving the existence of the intermediate region is not always easy so for our purposes we shall have to bypass the proof, but further study can be found in Ekhaus[11, 12] and MacGillivray[13]. With this new overlap domain we now define our intermediate variable as

\[
x_\eta = \frac{x}{\eta(\epsilon)}
\]

with \( \eta_0(\epsilon) < \eta(\epsilon) < \eta_1(\epsilon) \). In our example it is sufficient to take a simplified intermediate variable as

\[
x_\eta = \frac{x}{\epsilon^\alpha}
\]

where \( 0 < \alpha < 1 \).

But before making use of this new matching technique we must turn our attention to finding the second term for the outer and inner expansions. For the outer expansion look at (3.3) and on matching the \( O(\epsilon) \)
terms we get
\[ \frac{d^2y_0}{dx^2} + 2 \frac{dy_1}{dx} + 2y_1 = 0 \]
which, using \( y_0 \) from (3.5), has solution
\[ y_1 = -\frac{1}{2} e^{1-x} + Ae^{-x}. \]
From the boundary conditions we know \( y_1(1) = 0 \) (since \( y_0(1) = 1 \)) and can evaluate the constant \( A \) to give
\[ y_1(x) = \frac{1}{2} (1 - x) e^{1-x}, \]
and hence a two term expansion for the outer solution \( y(x) \) is now
\[ y(x) \sim e^{1-x} + \frac{1}{2} (1 - x) e^{1-x}. \] (3.16)

Now consider the original problem in terms of the inner expansion (3.13) and look at the terms of order \( O(1) \):
\[ \frac{d^2Y_1}{dX^2} + 2 \frac{dY_1}{dX} = -2Y_0. \]
The general solution to this is
\[ Y_1 = A + Be^{-2X} - X e^{1} - X e^{1-2X} \]
and using the boundary condition \( Y_1(0) = 0 \) we can remove one of the constants to give
\[ Y_1 = B(1 - e^{-2X}) - X e^{1}(1 + e^{-2X}). \]
So now we have a two term expansion for the boundary layer solution:
\[ Y(X) \sim e^{1} - e^{1-2X} + \epsilon(B(1 - e^{-2X}) - X e^{1}(1 + e^{-2X})) \] (3.17)
with \( B \) a constant, which must be evaluated by the matching process.

Now to match our two expansions together we want to write each in terms of the intermediate variable (3.15) and they should match. From (3.16) we get the outer expansion as
\[ y \sim e^{1-x} e^\alpha + \frac{1}{2} x e^\alpha e^{1-x}, \]
which after some tweaking using a Taylor expansion for \( e^{-x} e^\alpha \) gives, to two terms
\[ y \sim e^{1} - x e^{\alpha} e^{1} + \frac{1}{2} e^{1}. \]

From our inner expansion (3.17) we get, in terms of the new variable
\[ Y \sim e^{1} \left(1 - e^{\frac{-2x}{\epsilon - \beta}}\right) + \epsilon \left\{ B \left(1 - e^{\frac{-2x}{e^{1} - \epsilon^{1}}}ight) - \frac{x}{e^{1} - \epsilon} e^{1} \right\}. \]
Expanding the \{\} and noting that \( \frac{-2x}{e^{1} - \epsilon} \to \infty \) as \( \epsilon \to 0 \) we have
\[ Y \sim e^{1} - x e^{\alpha} e^{1} + B e^{1} \]
hence \( B = \frac{1}{2} e^{1} \) and our two term approximation within the boundary layer is
\[ Y(X) \sim e^{1} - e^{1-2X} + \epsilon \left(\frac{1}{2} e^{1}(1 - e^{-2X}) - X e^{1}(1 + e^{-2X})\right) \] (3.18)
As before the composite expansion is attained by adding (3.16) and (3.18) and subtracting the common limit to give as a final result
\[ y \sim e^{1-x} - (1 + x) e^{\frac{1}{2} x} + \frac{1}{2} \epsilon \left( e^{1} (1 - x) - e^{1-\frac{2x}{\epsilon}} \right) \] (3.19)
3.2. Summary. In a nutshell the method of matched asymptotic expansions can be useful for differential equations with an $\epsilon$ coefficient multiplying the highest order derivative. Usually these contain a boundary layer preventing the complete set of boundary conditions being satisfied by a regular perturbation solution. Where the regular solution fails we introduce new coordinates to describe the solution inside the boundary layer and produce two separate approximations valid over different sections of the domain. These solutions must be matched together and combined to form a single expansion valid universally.

3.3. Extensions. At the beginning of our problem it was identified that a boundary layer exists at either $x = 0$ or $x = 1$ and we assumed it was located at $x = 0$. Identifying the location of a boundary layer is one area where printed texts do not appear to give enough information. Murray[4] writes 'identifying the layer location comes from experience' while Van Dyke’s[6] method makes use of the exact solutions to the problems to determine the layer location - not much use for a problem that cannot easily be solved exactly.

Had the boundary layer been in another location we would need to use a more general boundary layer coordinate in (3.6). In general the transformation to use would be

$$X = \frac{x - x_0}{\delta(\epsilon)}$$

where $x_0$ represents the location of the layer.

In our example we had simple layer at one of the boundaries of the interval but much more complex layer dependence is possible. Layers can occur a random location in the middle of the domain which makes problems immediately more difficult as the location is not always easy to determine. It is even possible to have nested layers, where we discover that there are two possibilities in choosing $\delta(\epsilon)$ in the layer coordinate (3.6). One example of an equation resulting in nested boundary layers, from White[7], is the problem

$$\epsilon^3 x^2 \frac{d^2 y}{dx^2} + x^2 \frac{dy}{dx} - y(x^3 + \epsilon) = 0,$$

with $y(0) = 1$, $y(1) = \sqrt{e}$.

Boundary layer problems are amongst the most common of all asymptotic problems. The Navier-Stokes equations governing fluid flow at high Reynolds number are the typical example dealt with extensively by Van Dyke[6] and many other applied mathematicians. Examples can also be found in from Chemistry surrounding reaction-diffusion systems. More recent - and very much still developing - adaptations of singular perturbation theory have appeared in financial mathematics with some examples concerning small limits of volatility in dealing with the Black-Scholes option pricing model [10].

4. Method of Multiple Scales

A second type of singular perturbation problem fails not due to the loss of the leading order term, but instead these problems fail to be valid when the independent variable becomes large in the unbounded domain. Problems like this are common in systems dependent on time, thus an approximation found may be be valid initially but will deteriorate over time. As such they are also commonly known as time scale problems or often secular perturbation problems. In the previous section the method was to find two separate expansions over different regions and then match and combine them together into one composite expansion. Here the general idea is to start with a composite expansion and then split the independent variable into two separate scales, or regions, to filter out the offending terms in the expansion. This is illustrated in the example below.
4.1. Introductory Example. Consider the problem, featured in Murray[4]

\[
\frac{d^2y}{dt^2} + \epsilon \frac{dy}{dt} + y = 0
\]

\[
y(0) = a,
\]

\[
y'(0) = 0.
\]

(Using ‘ notation for the first time to indicate derivatives with respect to \( t \)).

Firstly we want to see why the singular nature arises so we will try and seek a regular expansion (\( y \sim y_0 + \epsilon y_1 + \epsilon^2 y_2 + \ldots \)). Substituting this into the differential equation and equating the coefficients of \( \epsilon \) in the usual manner gives

\[
y''_0 + y_0 = 0,
\]

\[
y''_1 + y_1 = -y_0
\]

for the first two terms of the expansion. The boundary conditions given in 4.1 tell us that \( y_0(0) = a \), \( y_1(0) = 0 \), \( y'_0(0) = 0 \) and \( y'_1(0) = 0 \). Using these we can find \( y_0 \) and \( y_1 \) to give an expansion to \( O(\epsilon) \):

\[
y \sim a \cos t + \frac{1}{2} \epsilon \cos t - \frac{1}{2} \epsilon^2 \cos t
\]

(4.2)

The difficulty here is the \( \epsilon t \cos t \) term which is known as a secular term. The solution is valid for small values of \( t \), where \( t = O(1) \) and \( \epsilon t \) is much smaller valued than the term independent of \( \epsilon \), but as \( t \) increases it eventually reaches a point where \( \epsilon t \) is no longer small in comparison. So when \( \epsilon t \) becomes as large as the first term, \( \epsilon t = O(1) \) or equivalently \( t = O(\frac{1}{\epsilon}) \), the second term is no longer smaller and in fact it continues to grow and invalidates the approximation.

Figure 3 shows the regular solution (4.2) plotted against the exact solution using MAPLE with \( \epsilon = 0.1 \) and \( a = 1 \) in the second initial condition. The exact solution is degenerative but due to the secular term...
the perturbation solution continues to oscillate with increasing amplitude. We see that the solution has reasonable accuracy until around \( t = O \left( \frac{1}{\epsilon} \right) \) (= 10), as expected.

Effectively the problem has different characteristics at two different time scales, one that is \( O(1) \) and another that is \( O(\frac{1}{\epsilon}) \) so the technique to solve the problem is to introduce two new variables:

\[
\begin{align*}
t_1 &= \epsilon t \\
t_2 &= t(1 + p_1 \epsilon^2 + p_2 \epsilon^3 + \ldots).
\end{align*}
\]

Notice that in \( t_2 \) there is no term of order \( \epsilon \) as this would result in \( t_1 \) being a parameter of \( t_2 \).

Using these new variables along with the chain rule we can find the differentiation terms we need to substitute the system into the differential equation in (4.1):

\[
\begin{align*}
\frac{d}{dt} &= \frac{dt_1}{dt} \frac{\partial}{\partial t_1} + \frac{dt_2}{dt} \frac{\partial}{\partial t_2} = \epsilon \frac{\partial}{\partial t_1} + (1 + p_1 \epsilon^2 + \ldots) \frac{\partial}{\partial t_2} \\
\frac{d^2}{dt^2} &= \epsilon^2 \frac{\partial^2}{\partial t_1^2} + 2\epsilon(1 + p_1 \epsilon^2 + \ldots) \frac{\partial^2}{\partial t_1 \partial t_2} + (1 + 2p_1 \epsilon^2 + \ldots) \frac{\partial^2}{\partial t_2^2}
\end{align*}
\]

The expansion for the solution will now be of the form

\[
y \sim y_0(t_1, t_2) + \epsilon y_1(t_1, t_2) + \epsilon^2 y_2(t_1, t_2) + \ldots \quad (4.4)
\]

To find this expansion we substitute (4.4) and (4.3) into the differential equation (4.1) and equate coefficients of \( \epsilon \), as we have done in previous examples so fewer details will be given here. We want a two-term approximation so are looking for the \( y_0 \) and \( y_1 \) terms but to obtain these we will discover we must find the \( y_2 \) also.

\[
\begin{align*}
O(1) : & \quad \frac{\partial^2 y_0}{\partial t_2^2} + y_0 = 0, \quad (4.5) \\
O(\epsilon) : & \quad \frac{\partial^2 y_1}{\partial t_2^2} + y_1 = -\frac{\partial y_0}{\partial t_2} - 2 \frac{\partial^2 y_0}{\partial t_2^2} \frac{\partial}{\partial t_1}
\end{align*}
\]

Initially the appearance of partial fractions may make it appear that we’ve complicated the problem but in fact the way they are arranged here, with only \( t_2 \) appearing on the left hand side, shows that \( t_2 \) is essentially a parameter. To solve the terms we also need the transform our boundary conditions into the new system of variables.

\[
\begin{align*}
y_0(0, 0) &= a, \quad \frac{\partial y_0}{\partial t_2}(0, 0) = 0 \\
y_1(0, 0) &= 0, \quad \frac{\partial y_1}{\partial t_2}(0, 0) + \frac{\partial y_0}{\partial t_1}(0, 0) = 0
\end{align*}
\]

Now we can solve the equations in (4.5) to find the \( y_0 \) term.

\[
y_0(t_1, t_2) = A_0(t_1) \cos t_2 + B_0(t_1) \sin t_2 \quad (4.7)
\]

where \( A_0 \) and \( B_0 \) are (at this point) unknown functions of \( t_1 \) satisfying the boundary conditions

\[
\begin{align*}
A_0(0) &= a, \quad B_0(0) = 0.
\end{align*}
\]

Now use (4.7) in the equation for \( y_1 \) from (4.5) to give

\[
\frac{\partial^2 y_1}{\partial t_2^2} + y_1 = (2 \frac{dA_0}{dt_1} + A_0) \sin t_2 - (2 \frac{dB_0}{dt_1} + B_0) \cos t_2 \quad (4.9)
\]

\[16\]
Looking at the differential equation in (4.9) we see that secular terms will arise due to the $\sin t$ and $\cos t$ terms unless we choose $A_0$ and $B_0$ in such a way that these terms vanish. So we want
\begin{align*}
2 \frac{dA_0}{dt_1} + A_0 &= 0 \quad \text{and} \\
2 \frac{dB_0}{dt_1} + B_0 &= 0.
\end{align*}

Solving these for $A_0$ and $B_0$ and using the boundary conditions from (4.8) we get:
\begin{align*}
A_0(t_1) &= ae^{-\frac{1}{2}t_1} \\
B_0(t_1) &= 0
\end{align*}

so now we can complete equation (4.7) to give us the first term in the expansion,
\begin{align*}
y_0(t_1,t_2) &= ae^{-\frac{1}{2}t_1} \cos t_2.
\end{align*}

Now with (4.10) we can simplify the equation for $y_1$ from (4.9) to give
\begin{equation}
\frac{\partial^2 y_1}{\partial t_2^2} + y_1 = 0
\end{equation}

which has the solution
\begin{align*}
y_1(t_1,t_2) &= A_1(t_1) \cos t_2 + B_1(t_1) \sin t_2.
\end{align*}

The boundary conditions for $y_1$ from (4.6) tell us that $y_1(0,0)=0$ so it is clear to see that
\begin{align*}
A_1(0) &= 0
\end{align*}

Also from (4.6) we know
\begin{align*}
\frac{\partial y_1}{\partial t_2}(0,0) + \frac{\partial y_0}{\partial t_1}(0,0) &= 0
\end{align*}

Using (4.12) this becomes
\begin{equation}
-A_1(0) \sin(0) + B_1(0) \cos(0) = \frac{1}{2}a
\end{equation}

and hence
\begin{align*}
B_1(0) &= \frac{1}{2}a
\end{align*}

Now we see that we can determine $A_1$ and $B_1$ in a similar way to what was done for $A_0$ and $B_0$, by choosing them in such a way to suppress the secular terms in the next term of the expansion sequence. So we now equate the $O(\epsilon^2)$ terms in our original equation to get the necessary equation and boundary conditions for $y_2$.
\begin{align*}
\frac{\partial^2 y_2}{\partial t_2^2} + y_2 &= -\frac{\partial y_1}{\partial t_2} - \frac{\partial y_0}{\partial t_1} - 2p_1 \frac{\partial^2 y_0}{\partial t_1 \partial t_2} - 2 \frac{\partial^2 y_1}{\partial t_1 \partial t_2} \\
y_2(0,0) &= 0, \\
\frac{\partial y_2}{\partial t_2}(0,0) + \frac{\partial y_1}{\partial t_1}(0,0) + p_1 \frac{\partial y_0}{\partial t_2}(0,0) &= 0.
\end{align*}

With our knowledge of $y_0$ and $y_1$ we can immediately begin to evaluate the first differential equation for $y_2$ in (4.17) to get
\begin{equation}
\frac{\partial^2 y_1}{\partial t_2^2} + y_2 = [2 \frac{dA_1}{dt_1} + A_1(t_1)] \sin t_2 - [2 \frac{dB_1}{dt_1} + B_1(t_1)] - \frac{1}{2} \ae^{-\frac{1}{2}t_1} + \frac{1}{4} \ae^{-\frac{1}{2}t_1} - 2p_1 \ae^{-\frac{1}{2}t_1} \cos t_2
\end{equation}
Rearranged in this order it is clear to see that to suppress the secular terms we need $A_1(t_1)$ and $B_1(t_1)$ to satisfy:

\[ 2 \frac{dA_1}{dt_1} + A_1(t_1) = 0, \]
\[ 2 \frac{dB_1}{dt_1} + B_1(t_1) = 2a(p_1 + \frac{1}{8})e^{-\frac{1}{2}t_1}. \]

Using the boundary condition (4.14) the solution to $A_1$ is obviously $A_1(t_1) = 0$. Solving for $B_1$ using (4.16) we get

\[ B_1 = a(p_1 + \frac{1}{8})t_1e^{-\frac{1}{2}t_1} + \frac{1}{2}ae^{-\frac{1}{2}t_1}. \]

We will now use these in our expression for $y_1$ thus far from 4.13 to get

\[ y_1(t_1, t_2) = \left[a(p_1 + \frac{1}{8})t_1e^{-\frac{1}{2}t_1} + \frac{1}{2}ae^{-\frac{1}{2}t_1}\right] \sin t_2. \]

Remember that in (4.4), the approximation for $y$, $y_1$ has an $\epsilon$ coefficient which gives rise to another secular term $\epsilon t_1$. In real time this is equivalent to $\epsilon^2 t$ and so we say $t$ is of order $\frac{1}{\epsilon}$. As before this can be suppressed because we still have an arbitrary $p_1$ variable inside our $y_1$ term, so by choosing $p_1 = -\frac{1}{8}$ we remove this term altogether and we now have a uniformly valid two-term approximation to our problem:

\[ y \sim ae^{-\frac{1}{2}t_1}[\cos t_2 + \frac{1}{2}\epsilon \sin t_2 + ...]. \]

Finally we put this in terms of our original variable given that $t_1 = \epsilon t$ and $t_2 = t(1 - \frac{1}{8}\epsilon^2 + ...) to obtain

\[ y \sim ae^{-\frac{1}{2}\epsilon t}[\cos t(1 - \frac{1}{8}\epsilon^2 + ...) + \frac{1}{2}\epsilon \sin t(1 - \frac{1}{8}\epsilon^2 + ...) + ...]. \]

**Figure 4.** Asymptotic Solution to $O(\epsilon)$

To analyse our solution the result (4.21) is plotted against the exact result in figure 4 using MAPLE (still with $\epsilon = 0.1, a = 1$). In the plotted region the solution is now so accurate it is difficult to distinguish between the two plots.
4.2. **Summary.** To summarize, the essentials of the method of multiple scales are to introduce two time scales: a fast one \( t_2 \) and a slow one \( t_1 \), and expand a regular perturbation solution in terms of these new coordinates. The secular terms found in each stage can be suppressed by equating the arbitrary functions from one term in the expansion with the next. Thus we have a single solution valid over the complete domain that can easily be expanded with lower order order terms where desired.

The specific version of the multiple scales method used in this example is known as the Lindstedt-Poincaré (or occasionally strained coordinates) method, where the two variables are chosen as \( \epsilon t_1 \) and \( t_2 \) (\( p_1 + \epsilon^2 p_2 + ... \)), but this is not the only variant of multiple scales. Holmes[3] gives a good example of a problem the same as (4.1) but with a slightly simpler boundary condition using the more primitive coordinates \( t_1 = t \) and \( t_2 = \epsilon t \).

There are many other variations of this method that all work on the same basic principles, it is even possible to have a transformation involving only one new variable, for example \( t = \bar{t} + \epsilon p_1 \bar{t} + ... \).

The example given and the name *time scale problems* might suggest this method is only applicable to problems where time is the independent variable but this is far from the case. Many of the problems that are solvable using the matched asymptotic expansions method are also solvable by multiple scales, and in the next example we shall now solve the same problem that was given by (3.1) in the previous section.

4.3. **Multiple Scales and Boundary Layers.** Consider again problem (3.1) that we solved in the previous section using matched asymptotic expansions,

\[
\epsilon \frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} + 2y = 0,
\]

where \( y(0) = 0 \), \( y(1) = 1 \).

Here in brief detail we will show how this can be solved using the multiple scales method. Change the coordinates to

\[
x_1 = \frac{x}{\epsilon},
\]

\[
x_2 = x.
\]

The \( x_1 \) term corresponds to the boundary layer coordinate so \( 0 \leq x_1 \leq \infty \) and \( x_2 \) represents the outer layer which satisfies \( 0 \leq x_2 \leq 1 \). Using the new coordinates the equation from (4.22) becomes

\[
\epsilon \left\{ \frac{1}{\epsilon^2} \frac{\partial^2 y}{\partial x_1^2} + \frac{2}{\epsilon} \frac{\partial y}{\partial x_1} + \frac{\partial^2 y}{\partial x_2^2} \right\} + 2 \left\{ \frac{1}{\epsilon} \frac{\partial y}{\partial x_1} + \frac{\partial y}{\partial x_2} \right\} + 2y = 0.
\]

After some rearrangement and multiplying by \( \epsilon \), so the largest coefficient is \( O(1) \), we have:

\[
\frac{\partial^2 y}{\partial x_1^2} + 2\epsilon \frac{\partial y}{\partial x_1 \partial x_2} + \epsilon^2 \frac{\partial^2 y}{\partial x_2^2} + 2 \frac{\partial y}{\partial x_1} + 2\epsilon \frac{\partial y}{\partial x_2} + 2y = 0.
\]

With the two variables the expansion will be of the regular form

\[
y \sim y_0(x_1, x_2) + \epsilon y_1(x_1, x_2) + \epsilon^2 y_2(x_1, x_2)....
\]

The next step as always is to substitute (4.24) into (4.23) and then equate the different orders in the resulting equation. For this example we only want to find a one-term approximation to illustrate the technique so we will need to find the terms of \( O(1) \) and \( O(\epsilon) \).

From the \( O(1) \) terms we get the general solution for \( y_0 \) in terms of two unknown constants:

\[
y_0 = A_0(x_2) + B_0(x_2)e^{-2x_1}
\]
Choosing these constants depends on the solution for \( y_1 \) as we need to avoid secular terms, so taking the \( O(\epsilon) \) terms from (4.23) with (4.24) gives the equation

\[
\frac{\partial^2 y_1}{\partial x_1^2} + 2 \frac{\partial y}{\partial x_1} - 2 \frac{\partial^2 y_0}{\partial x_1 \partial x_2} - 2 \frac{\partial y_0}{\partial x_2} - 2 y_0.
\]

Solving this gives the general solution in terms of our undetermined constants \( A_0 \) and \( B_0 \),

\[
y_1 = A_1(x_2) + B_1(x_2)e^{-2x_1} - \left( \frac{dA_0}{dx_2} + A_0 \right)x_1 + (B_0 - \frac{dB_0}{dx_2})x_1 e^{-2x_1}.
\]

Since \( x_1 \) is free to shoot off to infinity secular terms secular terms will clearly apply here unless we assert that

\[
\frac{dA_0}{dx_2} + A_0 = 0 \quad \text{and} \quad B_0 - \frac{dB_0}{dx_2} = 0
\]

We find that \( A_0 = e^{1-x_2} \) and \( B_0 = e^{1+x_2} \) and hence our expansion to \( O(1) \) in terms of the original variable \( x \) is

\[
y \sim e^{1-x} - e^{1+x-2x}\epsilon
\]

Compare this to the solution from (3.14)

\[
y \sim e^{1-x} - e^{1-2x}\epsilon
\]

and we see there is an extra \( x \) in the second exponential. Due to the \( \frac{x}{\epsilon} \) term the \( 1 + x + \frac{2x}{\epsilon} \) is small everywhere except in the boundary layer where \( x \) is of order \( \epsilon \). Hence any effect this \( x \) has on the result is to order \( \epsilon \), and since our solutions are to \( O(1) \) the two results are in fact asymptotically equivalent.

4.3.1. Comparison. It is not uncommon to be able to solve problems using either of the two different methods here, which naturally raises the question of which is better. Both require the introduction of new coordinates, which as we can see from the example are identical in either method. Multiple scales has the advantage that a single uniformly valid solution is produced directly, avoiding any difficulty of matching and we need only to analyze one single equation. The drawback is that the method relies on a system of partial differential equations which can become very large if we are looking to expand to a high order. It turns out that wherever we can apply multiple scales we also have the option of using matched asymptotic expansions, but not vice versa[6]. For that reason then matched asymptotic expansions could be considered the more fundamental and more dependable method.

5. Conclusion

To finish I think it is important to emphasize how small a portion of asymptotic analysis we have covered here. The scope for further study is monstrous, but that must not detract from the importance of what has been achieved. We have introduced the concepts from the original definitions of asymptotic analysis and gone forward to use them in examples from perturbation theory.

Perturbation theory is one of the most important methods of approximation due to its strong historical connection with physics, particularly the area of fluid mechanics. It is still as important today as it was during the development stages and in fact the previously mentioned examples in financial engineering confirm that new areas where perturbation techniques are useful are still to be found.

The two methods we met here, matched asymptotic expansions and multiple scales, are two of the most widely useful and fundamentally important, confirmed by the volume of texts that choosing to consider them both alongside a changing variety of others.
To extend our research in the areas we have crossed we could further investigate the method of matched asymptotic expansions. Illustrated was a simple case where layers existed at one of the boundaries and the layer was easy to locate. In reality there may be further complications where the location of a layer can be difficult to define or we can have several layers all present to complicate a single problem.

Additionally in this project we simplified our examples by considering only ordinary differential equations up to second order. The same methods used can be easily adapted to solve differential equations of much higher orders. With slightly trickier adaptation they can also be extended into partial differential equations, and this is vastly important in fluid dynamics.

Finally, looking back to the opening sentence suggests that asymptotic analysis can roughly be divided into two main forms. We investigated methods of approximating the solutions of differential equations. What we made no mention of was integral solutions. There are a wide range of methods that can be used to approximate integrals where necessary, an example of which is possibly the most basic of all asymptotic methods: integration by parts. More applicable examples include Laplace’s method, Steepest descents and Stationary phase - all of which are introduce in Murray[4]. And there should be no surprise that these two areas are not entirely disjoint. There are many methods for solving differential equations that produce a solution defined by an integral so in fact these integral solutions can become vastly important to the types of problem we studied here.

REFERENCES